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Resonances in the Rigged Hilbert Space and Lax–Phillips Scattering Theory

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The rigged Hilbert space formalism of quantum mechanics provides a framework in which one can identify resonance states and obtain the typical exponential decay law. However, there remain questions of the interpretation and extraction of physical information through the calculation of expectation values of observables. The Lax–Phillips scattering theory provides a mathematical construction in which resonances are assigned with states in a Hilbert space, thus no such difficulties arise. The original Lax–Phillips structure is inapplicable within standard nonrelativistic quantum theory. Through the powerful theory of H^p spaces certain relations between the two theories are uncovered, which suggest that a search for a "unifying" framework might prove useful.

KEY WORDS: resonance; semigroup evolution; Lax–Phillips theory; rigged Hilbert space; H^P spaces.

1. INTRODUCTION

Recent years have seen a rising interest in the rigged Hilbert space (RHS) description of quantum mechanical resonances. It is found that within the RHS structure $\Phi \subset H \subset \Phi^\times$ (where H is the quantum mechanical Hilbert space) utilized by this theory it is possible to assign to a quantum mechanical resonance a state in $\Phi^\times$, the "larger" sector of the Gel'fand triplet (Bailey and Schieve, 1978; Baumgartel, 1975; Bohm, 1986; Bohm *et al.*, 1989; Bohm and Gadella, 1989; Horwitz and Sigal, 1978; Parravicini *et al.*, 1980). This state then exhibits a semigroup evolution law under the evolution generated by the extension $\mathbf{H}^\times$ of the Hamiltonian to $\Phi^\times$. In particular, one obtains the typical exponential decay of the time evolution of the resonance. These properties, which are unattainable within the standard quantum mechanical Hilbert space formulation of the problem, via the Wigner–Weisskopf model (Weisskopf and Wigner, 1930), renders the model for resonances thus obtained particularly appealing. However, the representation

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of the resonance obtained in the RHS formalism is in a Banach space which does not coincide with the quantum mechanical Hilbert space, and does not have the properties of the Hilbert space, such as the existence of a scalar product and the possibility of calculating expectation values. One cannot compute physical properties other than the lifetime in this way.

The Lax–Phillips scattering theory (Lax and Phillips, 1967), originally developed for the description of resonances in electromagnetic or acoustic scattering phenomena, has been used as a framework for the construction of a description of irreversible resonant phenomena in the quantum theory (this is referred to as the quantum Lax–Phillips theory). As in the RHS formalism, this theory leads to a time evolution of resonant states which is of semigroup type, exhibiting the exponential decay law for the resonance. In principle the quantum Lax–Phillips theory (Eisenberg and Horwitz, 1997; Horwitz and Piron, 1993; Strauss and Horwitz, 2000a,b) provides the possibility of constructing a fundamental theoretical description of the resonant system. The utilization of the mathematical structure of the original Lax–Phillips theory ensures that the resonance is represented as a state in a Hilbert space. In this way one retains the interpretation of the quantum mechanical state of the system, enabling the calculation of expectation values of observables.

Although the quantum Lax–Phillips theory was found to provide a proper description for nonrelativistic open quantum systems and certain relativistic quantum mechanical models, the fact that the theory necessitates a generator of evolution with a spectrum which is unbounded from below, makes it inapplicable within the framework of standard nonrelativistic quantum mechanics.

In this work I consider the RHS model for quantum mechanical resonances developed by A. Bohm and M. Gadella (Bohm, 1979a,b, 1980, 1981; Bohm and Gadella, 1989; Gadella, 1983a,b, 1984). It is my purpose to show that a close examination of the mathematical structures of the RHS model for resonances and the Lax–Phillips scattering theory reveals a strong relation in their description of resonances. In a sense one can say that the RHS model "reaches out" into the Lax–Phillips Hilbert space to obtain the desired exponential decay, semigroup evolution, and its connection to poles of the S-matrix in the second Riemann sheet. These observations bring up the possible existence of a "unifying" framework which inherently includes the advantages of both of the theories with which we are concerned here. Such framework has been proposed and the full consequences of its application to the description of resonances are explored elsewhere (Strauss and Horwitz, in preparation).

The outline of the rest of this paper is as follows. In section 2, I describe the Lax–Phillips scattering theory. Section 3 provides some of the broader mathematical context giving rise to the structure of the Lax–Phillips scattering theory. In section 4, I essentially repeat the construction of the RHS model for resonances. In the process, key parts of the theory are reformulted in terms of the fundamental structures found in the Lax–Phillips scattering theory. Thus, it is seen that the mechanism of identification of the resonance states and their relation to singularities of the S-matrix in the RHS model and in the Lax–Phillips scattering theory are strongly related. Conclusions follow in section 5.

2. THE LAX–PHILLIPS SCATTERING THEORY

The Lax–Phillips scattering theory was originally developed by P. D. Lax and R. S. Phillips for the description of the process of scattering of electromagnetic waves or acoustic waves off a spatially bounded target (Lax and Phillips, 1967). The theory, devised for application to hyperbolic partial differential equations (wave equations), is based on a Hilbert space description of the propagating waves. The time evolution of these waves is given by a unitary evolution group of operators defined on the Hilbert space, satisfying several conditions. In the framework thus developed there arises in a natural way a semigroup defined on a subspace of the Hilbert space corresponding to waves interacting with the target. One of the main results of the Lax–Phillips scattering theory is the fact that the eigenvalues of the generator of the semigroup are associated with poles of the S-matrix of the theory. It is the existence of the semigroup and the relation between eigenvalues of its generator and poles of the S-matrix which renders the mathematical framework of the Lax–Phillips scattering theory particularly attractive for adaptation to the description of resonances in quantum theory.

Consider a Hilbert space H and an evolution group of unitary operators $\mathbf{U}(\mathrm{t})$ on H. Suppose that there exists two distinguished subspaces D_- and D_+ which have the properties that D_- is orthogonal to D_+ and

$$\begin{aligned}
&\mathbf{U}(t)D_- \subset D_-, \qquad t \le 0\\
&\mathbf{U}(t)D_+ \subset D_+, \qquad t \ge 0\\
&\cap_t \mathbf{U}(\mathbf{t})D_\pm = \{0\}\\
&\overline{\cup_t \mathbf{U}(t)D_\pm} = H
\end{aligned} \tag{2.1}$$

We call D_- the *incoming subspace* and D_+ the *outgoing subspace*. D_- corresponds to incoming waves which do not interact with the traget prior to $t = 0$ and D_+ corresponds to outgoing waves which do not interact with the target after $t = 0$. These properties are reflected in the stability properties of D_- and D_+ in Eq. (2.1).

Let $L^2(-\infty, \infty; N)$ be the space of L^2 functions defined on $(-\infty, \infty)$ and taking their values in a Hilbert space N. Ja. G. Sinai (Cornfield *et al.*, 1982) proved that if the conditions of Eq. (2.1) hold for the outgoing space D_+ then the following theorem holds.

Theorem A *(Ja. G. Sinai). if D_+ is an outgoing subspace with respcet to a group of unitary operators* ***U(t)****, then H can be represented isometrically as the Hilbert space of functions $L^2(-\infty, +\infty; N)$ for some auxiliary Hilbert space N*

so that **U(t)** *goes into translation to the right by t units, and* D_+ *is mapped onto* $L^2(0, +\infty; N)$. *This representation is unique up to an isomorphism of N.*

A representation of this kind is called *outgoing translation representation* for the group **U**(t). An analogous representation theorem holds for an incoming subspace D_-, i.e., there is a representation in which H is mapped onto the Hilbert space $L^2(-\infty, +\infty; N)$, D_- is mapped onto $L^2(-\infty, 0; N)$, and **U(t)** acts as translation to the right by t units. This representation is called the *incoming translation representation*.

For most purposes it is more convenient not to work with the translation representations but with two different representations, called spectral representations. By Fourier transformation of the incoming translation representation and the outgoing translation representation we obtain the *incoming spectral representation* and *outgoing spectral representation* respectively. According to the Paley–Wiener theorem (Paley and Wiener, 1934), in the incoming spectral representation, the subspace D_- is represented by the Hilbert space of functions $H_N^+(R)$ consisting of boundary values on R of functions in the Hardy space $H_N^2(\prod)$. Denoting the upper half of the complex plane by $\prod$, the space $H_N^2(\prod)$ is characterized as the space of analytic vector valued functions on $\prod$, taking their values in the auxiliary Hilbert space N, and such that

$$\sup_{y>0} \int_{-\infty}^{+\infty} \| f(x+iy)\|_N^2 \, dx < C$$

For some constant $C > 0$. In the outgoing spectral representation the subspace D_+ is represented, according to the Paley-Wiener theorem, by the function space $H_N^-(R)$ consisting of boundary values of functions in $H_N^2(\underline{\prod})$, a Hardy space of vector valued functions (taking values in N) on the lower half-plane $\underline{\prod}$.

Let $\mathbf{W}_+$ and $\mathbf{W}_-$ denote the operators that map elements of H to their outgoing, respectively incoming, translation representers. We call the operator

$$\mathbf{S}^{\text{L.P.}} \equiv \mathbf{W}_+ \mathbf{W}_-^{-1} \tag{2.2}$$

the *abstract scattering operator* associated with the group **U(t)** and the pair of spaces D_- and D_+. It was proved by Lax and Phillips that $\mathbf{S}^{\text{L.P.}}$ is equivalent to the standard definition of the scattering operator. The abstract scattering operator has the following properties:

a) $\mathbf{S}^{\text{L.P.}}$ is unitary;
b) $\mathbf{S}^{\text{L.P.}}$ commutes with translations; and
c) $\mathbf{S}^{\text{L.P.}}$ maps $L^2(-\infty, 0; N)$ into itself.

Property (b) is due to the fact that $\mathbf{S}^{\text{L.P.}}$ is a map between two translation representations. One can understand property (c) by noting that in the incoming translation representation the subspace D_- is identified with the space of functions

$L^2(-\infty, 0; N)$ and in the outgoing translation representation D_+ is represented as $L^2(0, +\infty; N)$. The orthogonality of D_- and D_+ then implies that in the outgoing representation D_- is represented by a subspace of $L^2(-\infty, 0; N)$ and property (c) above follows.

Going over to the spectral representation the scattering operator transforms into $\mathcal{S}^{\text{L.P.}} \equiv \mathbf{F}\mathbf{S}^{\text{L.P.}}\mathbf{F}^{-1}$, where $\mathbf{F}$ is the Fourier trasform operator. Properties (a)–(c) above then imply corresponding properties for $\mathcal{S}^{\text{L.P.}}$:

a$'$) $\mathcal{S}^{\text{L.P.}}$ is unitary.
b$'$) $\mathcal{S}^{\text{L.P.}}$ commutes with multiplication by scalar functions.
c$'$) $\mathcal{S}^{\text{L.P.}}$ maps $H_N^+(R)$ into itself.

According to a special case of a theorem of Foures and Segal (1955), an operator satisfying properties (a')–(c') can be realized as a multiplicative, operator-valued function $\mathcal{S}(\sigma)$ (with $\sigma \in R$), mapping N into N for each $\sigma \in R$ and satisfying

a$''$) $\mathcal{S}(\sigma)$ is the boundary value of an operator valued function $\mathcal{S}(z)$ analytic for $\text{Im}\, z > 0$;
b$''$) $\|\mathcal{S}(z)\|_N \leq 1$ for $\text{Im}\, z > 0$; and
c$''$) $\mathcal{S}(\sigma)$, $\sigma \in R$ is, pointwise, a unitary opertor on the auxiliary Hilbert space N.

Lax and Phillips define a family of operators $\{\mathbf{Z}(t); t \geq 0\}$ by

$$\mathbf{Z}(t) \equiv \mathbf{P}_+\mathbf{U}(t)\boldsymbol{P}_-, \qquad t \geq 0 \tag{2.3}$$

Here $\mathbf{P}_+$ is the orthogonal projection of H onto the orthogonal complement of D_+ and $\mathbf{P}_-$ is the orthogonal projection of H onto the orthogonal complement of D_-. From the definition Eq. (2.3) it is evident that (for any t) $\mathbf{Z}(t)$ annihilates D_- and its range is orthogonal to D_+. For any element $f \in D_+$ and $t \geq 0$ we have, using the stability properties of D_+ from Eq. (2.1)

$$\mathbf{Z}(t) f = \mathbf{P}_+\mathbf{U}(t)\mathbf{P}_- f = \mathbf{P}_+\mathbf{U}(t) f = 0$$

hence the subspace D_+ is annihilated by $\mathbf{Z}(t)$. Furthermore, for any $f \in D_-$ and any $g \in H$ we have

$$\begin{aligned}(f, \mathbf{Z}(t)g)_H &= (f, \mathbf{P}_+\mathbf{U}(t)\mathbf{P}_-g)_H = (\mathbf{P}_+ f, \mathbf{U}(t)\mathbf{P}_-g)_H \\ &= (\mathbf{U}^\dagger(t) f, \mathbf{P}_-g)_H = (\mathbf{P}_-\mathbf{U}(-t) f, g)_H\end{aligned} \tag{2.4}$$

The stability properties of D_-, Eq. (2.1), and the definition of $\mathbf{P}_-$ then implies that

$$(f, \mathbf{Z}(t)g)_H = 0, \qquad f \in D_-$$

and we find that D_- is not in the range of $\mathbf{Z}(t)$. We conclude that the family $\{\mathbf{Z}(t); t \geq 0\}$ annihilate D_- and D_+ and take the subspace $K = H \ominus (D_- \oplus D_+)$ into itself.

It is easily proved that the family of operators $\{\mathbf{Z}(t); t \geq 0\}$ forms a continuous semigroup. Considering a vector $f \in K$, we have

$$\begin{aligned}\mathbf{Z}(t_1)\mathbf{Z}(t_2) f &= \mathbf{P}_+\mathbf{U}(t)\mathbf{P}_-\mathbf{Z}(t_2) f = \mathbf{P}_+\mathbf{U}(t)\mathbf{Z}(t_2) f \\ &= \mathbf{P}_+\mathbf{U}(t_1)\mathbf{P}_+\mathbf{Z}(t_2) f, \qquad t_1, t_2 \geq 0\end{aligned} \tag{2.5}$$

The stability properties of the subspace D_+, Eq. (2.1), imply the following identity:

$$\mathbf{P}_+\mathbf{U}(t)(\mathbf{I} - \mathbf{P}_+) = 0, \quad t \geq 0 \tag{2.6}$$

Inserting this identity into the previous equation, we find

$$\begin{aligned}\mathbf{Z}(t_1)\mathbf{Z}(t_2) f &= \mathbf{P}_+\mathbf{U}(t_1)\mathbf{P}_+\mathbf{U}(t_2) f = \mathbf{P}_+\mathbf{U}(t_1)[(\mathbf{I} - \mathbf{P}_+) + \mathbf{P}_+]\mathbf{U}(t_2) f \\ &= \mathbf{P}_+\mathbf{U}(t_1)\mathbf{U}(t_2) f = \mathbf{P}_+\mathbf{U}(t_1 + t_2)\mathbf{P}_- f \\ &= \mathbf{Z}(t_1 + t_2) f, \qquad t_1, t_2 \geq 0, \quad f \in K\end{aligned} \tag{2.7}$$

Lax and Phillips prove the following theorem, providing further properties of the semigroup $\{\mathbf{Z}(t); t \geq 0\}$.

Theorem B. *The operators* $\{\mathbf{Z}(t); t \geq 0\}$ *annihilate* D_+ *and* D_-*, map the orthogonal complement* $K = H \ominus (D_- \oplus D_+)$ *into itself, and form a strongly continuous semigroup (i.e.,* $\mathbf{Z}(t_1)\mathbf{Z}(t_2) = \mathbf{Z}(t_1 + t_2)$*) of contraction operators on* K*. Furthermore,* $\mathbf{Z}(t)$ *tends strongly to zero as* $t \to \infty$*:* $\lim_{t\to\infty} \mathbf{Z}(t)x = 0$ *for every* x *in* K*.*

Henceforth, the family of operators $\{\mathbf{Z}(t); t \geq 0\}$ will be called the Lax–Phillips semigroup. The operator-valued function $\mathcal{S}(z)$, with properties $(a'') - (c'')$ above, will be called the Lax–Phillips S-matrix. Properties $(a'') - (c'')$ characterize the Lax–Phillips S-matrix as an analytic function on the upper half-plane. The analytic continuation of $\mathcal{S}(z)$ to the lower half-plane is given by

$$\mathcal{S}(z) \equiv [\mathcal{S}^\dagger(\bar{z})]^{-1}, \qquad \text{Im}\, z < 0 \tag{2.8}$$

One of the main results of the Lax–Phillips scattering theory is the following theorem proved by Lax and Phillips.

Theorem C. *Let* $\mathbf{B}$ *denote the generator of the semigroup* $\mathbf{Z}(t)$*. If Im* $\mu < 0$*, then* μ *belongs to the point spectrum of* $\mathbf{B}$ *if and only if* $\mathcal{S}^\dagger(\bar{\mu})$ *has a nontrivial null space.*

This theorem establishes a very important relation between the eigenvalues of the generator $\mathbf{B}$ of the Lax–Phillips semigroup and poles of the analytic continuation of the Lax–Phillips S-matrix $\mathcal{S}(z)$ to the lower half-plane. This theorem provides a motivation for the use of the framework of the Lax–Phillips scattering theory for the description of quantum mechanical resonances. It enables the possibility of associating certain, well-defined, vectors in a Hilbert space H, with resonance poles of the S-matrix, such that these vectors are eigenvectors of an evolution semigroup related to the unitary evolution group on H.

The proof of the theorem above is illuminating and is referred to again later on. Therefore, we reproduce it here (Lax and Phillips (1967) provide both this version and a second version of the proof in their book.

Proof: Let x be an eigenvector of the generator $\mathbf{B}$ of the Lax–Phillips semigroup with eigenvalue μ

$$\mathbf{B}x = \mu x \tag{2.9}$$

then

$$i\frac{d}{dt}\mathbf{Z}(t)x = \mathbf{B}\mathbf{Z}(t)x = \mathbf{Z}(t)\mathbf{B}x = \mu\mathbf{Z}(t)x$$

and so

$$\mathbf{Z}(t)x = e^{-i\mu t}x, \qquad t \geq 0 \tag{2.10}$$

As indicated above the domain of $\mathbf{B}$ contains vectors in $K \subset H$. In the outgoing representation the Hilbert space H is represented by the function space $L^2(-\infty, +\infty; N)$ which we take to be functions of the real parameter s ($s \in R$). Vectors in K are then represented by continuous functions in $L^2(-\infty, 0; N)$ and are supported on $s \leq 0$. Let f represents, in this representation, an eigenvector $x \in K$ of $\mathbf{B}$. In the outgoing translation representation Eq. (2.10) is transformed into

$$f(s-t) = e^{-i\mu t} f(s), \qquad s \leq 0, \quad t \geq 0 \tag{2.11}$$

we set $s = 0$ and find

$$f(-t) = e^{-i\mu t} f(0), \qquad t \geq 0$$

Denote $n = f(0)$ and $s = -t$, then

$$f(s) = \begin{cases} e^{i\mu s}n & s \leq 0 \\ 0 & s > 0 \end{cases} \tag{2.12}$$

Equation (2.12) is the general form for the outgoing translation representer of an eigenvector of $\mathbf{B}$. Now, in the incoming translation representation the subspace $D_- \subset H$ is represented by $L^2(-\infty, 0; N)$. When mapped to the outgoing translation representation, this subspace is represented by $\mathbf{S}^{\text{L.P.}}L^2(-\infty, 0; N) \subset L^2(-\infty, 0; N)$. The orthogonality $K \perp D_-$ then imply that in the outgoing translation represention we have

$$(f, \mathbf{S}^{\text{L.P.}}k)_{L^2_N(R)} = 0, \qquad k \in L^2(-\infty, 0; N) \tag{2.13}$$

where f is, as above, an outgoing translation representer of $x \in K$. In the outgoing spectral representation we obtain

$$(\hat{f}, \mathcal{S}^{\text{L.P.}}\hat{k})_{L^2_N(R)} = 0 \tag{2.14}$$

where $\hat{k}$, $\hat{f}$ are the outgoing spectral representers corresponding to k and f respectively (below, functions and operators in the outgoing spectral representation are denoted with a hat), and $\mathcal{S}^{\text{L.P}}$ is the Lax–Phillips S-matrix in the spectral representation. Fourier transforming Eq. (2.12) we obtain

$$\hat{f}(\sigma) = i\frac{n}{\sigma - \mu}, \qquad \sigma \in R \tag{2.15}$$

Using Eq. (2.15), we write explicitly the scalar product in Eq. (2.14)

$$0 = (\hat{f}, \mathcal{S}^{\text{L.P.}}\hat{k})_{L^2_N(R)} = \int_{-\infty}^{+\infty} d\sigma\, \frac{(n, \mathcal{S}(\sigma)\hat{k}(\sigma))_{\mathbf{N}}}{\sigma - \bar{\mu}} \tag{2.16}$$

Since the vector-valued function $\mathcal{S}(z)\hat{k}(z)$ is a Hardy class function, it is possible to calculate the integral in Eq. (2.16) as a contour integral in the upper half-plane. The only contribution comes from the pole at $\bar{\mu}$ with the result

$$(n, \mathcal{S}(\bar{\mu})\hat{k}(\bar{\mu}))_N = 0 \tag{2.17}$$

Furthermore, the vector $k \in D_-$ is arbitrary and we conclude that $\mathcal{S}(\bar{\mu})$ has a positive codimension. This implies that $\mathcal{S}^\dagger(\bar{\mu})$ has a nontrivial null space and the proof is complete. □

This concludes the description of the scattering theory developed by Lax and Phillips. It turns out, though, that in order to complete the analysis of physical models treated within the framework given here, one needs to put the Lax–Phillips scattering theory into a more general mathematical context. This is also required for the purpose of including into the range of applicability of the theory classes of physical models for which this framework cannot be applied directly. Some aspects of this, more general, theory are discussed in the next section.

3. MATHEMATICAL BASIS OF THE LAX–PHILLIPS SCATTERING THEORY

Given a Lax–Phillips structure with an evolution group $\mathbf{U}(t)$ on the Hilbert space H, consider the family of operators $\{\mathbf{T}(t)\}_{t\geq 0}$ such that $\mathbf{T}(t) : H \to H$, $t \geq 0$, and

$$\mathbf{T}(t) \equiv \mathbf{P}_+\mathbf{U}(t), \qquad t \geq 0 \tag{3.1}$$

where, as in section 2 above, $\mathbf{P}_+$ is the projection on the orthogonal complement subspace to D_+ in H. Each element of this family of operators annihilates the subspace D_+, as can be seen, for example, from Eq. (2.1). Furthermore, noting that in Eq. (2.7) it is possible to replace $\mathbf{Z}(t_1)$, $\mathbf{Z}(t_2)$, and $\mathbf{Z}(t_1+t_2)$ by $\mathbf{T}(t_1)$, $\mathbf{T}(t_2)$, and $\mathbf{T}(t_1+t_2)$, respectively, and the vector $f \in K$ by any vector $f \in D_- \oplus K$, we find for any vector $f \in D_- \oplus K$

$$\mathbf{T}(t_1)\mathbf{T}(t_2) f = \mathbf{T}(t_1+t_2) f, \qquad t_1, t_2 \geq 0 \tag{3.2}$$

and so the family $\{\mathbf{T}(t); t \geq 0\}$ forms a one-parmeter semigroup. Finally, we observe from Eq. (2.5) that, for $f \in K$, we have

$$\mathbf{Z}(t_1)\mathbf{Z}(t_2) f = \mathbf{P}_+\mathbf{U}(t_1)\mathbf{P}_+\mathbf{U}(t_2) f = \mathbf{T}(t_1)\mathbf{T}(t_2) f, \quad t_1, t_2 \geq 0, \ \ f \in K \tag{3.3}$$

where $\{\mathbf{Z}(t); t \geq 0\}$ is the Lax–Phillips semigroup.

Consider now the Lax–Phillips outgoing translation representation. Denote the outgoing translation representer of an operator $\mathbf{T}(t)$ from the family defined in Eq. (3.1) by $\tilde{T}(t)$. Given any vector-valued function $f \in L^2(-\infty, +\infty; N)$ in the outgoing translation representation we have

$$(\tilde{T}(t) f)(s) = \begin{cases} f(s-t) & s \leq 0 \\ 0 & s > 0 \end{cases}, \qquad t \geq 0 \tag{3.4}$$

Denote the generator of the semigroup $\{\mathbf{T}(t); t \geq 0\}$ by $\mathbf{B}'$. Repeating the steps taken in Eqs. (2.9)–(2.12), with $\mathbf{T}(t)$ replacing $\mathbf{Z}(t)$, we find the spectrum of $\mathbf{B}'$ to be $\sigma(\mathbf{B}') = \{\mu | \operatorname{Im}\mu < 0\}$. If the outgoing translation representer of $\mathbf{B}'$ is denoted $\tilde{B}'$, then the eigenfunctions of $\tilde{B}'$ are given by

$$f_{\mu,n}(s) = \begin{cases} e^{i\mu s} n & s \leq 0 \\ 0 & s > 0 \end{cases}, \qquad \forall \mu, \operatorname{Im}\mu < 0, \quad \forall n \in N \tag{3.5}$$

where $f_{\mu,n}$ is an eigenfunction of $\tilde{B}'$ with eigenvalue μ. We find the representation of the semigroup $\{\mathbf{T}(t); t \geq 0\}$ and the eigenfunctions $f_{\mu,n}$ in the outgoing spectral representation. For this we need the definition of a Toeplitz operator on the Hardy space $H^+_N(R)$ (see, for example Rosenblum and Rovnyak, 1985, and references therein).

Definition (Toeplitz opertor on $H^+_N(R)$). Let $W \in L^\infty_{\mathcal{B}(N)}(R)$($\mathcal{B}(N)$ is the space of bounded linear operators on N). Let P_+ denote the projection of $L^2_N(R)$ on $H^+_N(R)$. Then the operator $T_W : H^+_N(R) \to H^+_N(R)$ defined by

$$T_W f = P_+ W f, \qquad f \in H^+_N(R) \tag{3.6}$$

is called a Toeplitz operator (on $H^+_N(R)$) with symbol W. Here W is the operator of pointwise multiplication by W, i.e., $(Wf)(\sigma) = W(\sigma) f(\sigma)$, $\sigma \in R$. We define the following multiplicative operator

$$[(\widehat{e^{-iEt}}) f](E) = e^{-iEt} f(E), \qquad f \in L^2_N(R), \quad E \in R \tag{3.7}$$

and, taking the Fourier transform of Eq. (3.4) and using the definition, Eq. (3.6), we find in the outgoing spectral representation

$$\hat{T}(t) f = P_+ \widehat{e^{-iEt}} f = T_{e^{-iEt}} f, \qquad f \in H_N^+(R) \tag{3.8}$$

and $\hat{T}(t) f = T_{e^{-iEt}} f = 0$ for $f \in H_N^-(R)$. The semigroup $\{\mathbf{T}(t); t \geq 0\}$ is, therefore, represented in the outgoing spectral representation, by the Toeplitz operator with symbol e^{-iEt}. Taking the Fourier transform of Eq. (3.5), we find that in the outgoing spectral representation the eigenfunctions of $\hat{T}(t) = T_{e^{-iEt}}$ are given by

$$\hat{f}_{\mu,n}(\sigma) = i \frac{n}{\sigma - \mu} \qquad \forall \mu, \operatorname{Im} \mu < 0, \quad \forall n \in N \tag{3.9}$$

Returning to Eq. (3.1) and (3.3), we identify the Lax–Phillips semigroup $\{\mathbf{Z}(t); t \geq 0\}$ as the restriction of $\{\mathbf{T}(t); t \geq 0\}$ to the subspace K, i.e.,

$$\mathbf{Z}(t) = \mathbf{T}(t)|K \tag{3.10}$$

where

$$K = H \ominus (D_- \oplus D_+) \tag{3.11}$$

The Lax–Phillips S-matrix, mapping the incoming representation onto the outgoing representation, is, from the mathematical point of view, a map of $L^2(-\infty, +\infty; N)$ onto itself. This map is characterized by its action on $H_N^+(R) \subset L^2(-\infty, +\infty; N)$ (which represents in the incoming spectral representation the subspace D_-) as an inner function (see secton 4 below). We have, therefore, $\mathcal{S}^{\text{L.P.}} = S_{\text{in}}$ for some inner function S_{in}. The mapping of $H_N^+(R)$ by the Lax–Phillips S-matrix results, therefore, in the subspace $S_{\text{in}} H_N^+(R)$ (representing D_- in the outgoing representation). The subspace $S_{\text{in}} H_N^+(R)$ is an invariant subspace for the action of the translations defined in Eq. (3.7) for $t < 0$ (this is the stability property of D_-, since the evolution is represented by translation). From Eq. (3.10) we infer that in the outgoinging spectral representation the Lax–Phillips semigroup is represented by the restriction of the semigroup in Eq. (3.8) to the subspace

$$\hat{K} = H_N^+(R) \ominus S_{\text{in}} H_N^+(R) \tag{3.12}$$

that is

$$\hat{Z}(t) = \hat{T}(t)|\hat{K} = T_{e^{-iEt}}|\hat{K} \tag{3.13}$$

We see that, in the outgoing spectral representation, the Lax–Phillips semigroup is given by the restriction of the semigroup of Eq. (3.8) to the orthogonal complement in $H_N^+(R)$ of an invariant subspace obtained by the action of an inner function S_{in} on $H_N^+(R)$. The main theorem of the Lax–Phillips theory (Theorem C of section 2) states that the eigenvalues of the restricted semigroup in Eq. (3.13) are related to the positive codimension of the inner (operator-valued) function $S_{\text{in}}(z)$ at certain points of the upper half-plane (or poles of the analytic continuation of $S_{\mathrm{in}}(z)$ into the lower half-plane).

We note that the broader mathematical context for the structure just described is within the Sz. Nagy–Foias theory of contraction operators on Hilbert spaces (Sz.-Nagy and Foias, 1970). In particular, the Lax–Phillips semigroup is related to the so-called compression of a shift (for a very thorough treatment of this type of operators, see Nikol'skii, 1986).

In the following section the RHS theory of resonances and its relation to the mathematical structure of the Lax–Phillips scattering theory is discussed in the context of purely spectral models. This simplifies the exposition of the main ideas due to the fact that in this case one has to deal only with scalar functions. It should be emphasized that the structures below can be recast into the general form of vector-valued functions.

4. RHS MODEL OF RESONANCES AND THE MATHEMATICAL FRAMEWORK OF THE LAX–PHILLIPS THEORY

4.1. Quantum Mechanics in the Rigged Hilbert Space

In the framework of quantum mechanics one assigns to the quantum mechanical system, at each point of time, a state vector in an appropriately constructed Hilbert space H. The physically observable quantities are associated with self-adjoint operators on H. To complete the description of the physics involved with the quantum mechanical system, an evolution law is supplied in the form of a suitable equation of evolution (Schrödinger equation in the case of nonrelativistic quantum theory).

The RHS approach to quantum mechanics is based on the observation that a Hilbert space constructed for the description of a typical quantum mechanical system is in a sense both "too big" and "too small." Such a Hilbert space inevitably contains "irrelevant" state vectors. There are vectors related to operators which are observables but which are not physically realizable and, furthermore, there are self-adjoint operators on H which one would want to associate with observables but which are unrealizable as physical observables for certain quantum mechanical systems. Examples are vectors which lie outside the domain of definition of an observable represented by an unbounded operator (e.g., infinite energy states, infinite momentum states, etc.) or the position operator for a bound state of an atom. In addition there are objects which are important in quantum theory and cannot be included within the standard framework. An important class of objects which fall into the later category are quantum mechanical resonances, with which this work is concerned.

With the above observations in mind the RHS formalism is centered on the construction of a Gel'fand triple (Gel'fand and Vilenkin, 1964) $\Phi \subset H \subset \Phi^{\times}$, where H is the quantum mechanical Hilbert space corresponding to the particular system considered. The nuclear space Φ is dense in H and is endowed with a topology τ_Φ which is finer than the norm topology inherited from H, and the space $\Phi^\times$ is the space of τ_Φ continuous antilinear functionals on Φ. As a result of the application of the Gel'fand–Maurin *nuclear spectral theorem*, the space Φ has the property that for every e.s.a. (essentially self-adjoint) operator on H there exists a complete set of generalized eigenvectors and one can implement in a rigorous way the Dirac formalism. The space Φ also has the property that all algebraic operations with operators are allowed and no questions of domains of definition arise. With respect to the topology τ_Φ on Φ the operators corresponding to physical observables form an algebra of continuous operators.

The fact that, in the RHS formalism, the Hamiltonian $\mathbf{H}$ is e.s.a. on H (i.e., with respect to the norm topology on H) and continuous with respect to the topology of Φ enables us to utilize the Nuclear Spectral, or Gelfand-Maurin, theorem (Gel'fand and Vilenkin, 1964). This guarantees the existence of a complete set of generalized eigenvectors belonging to $\Phi^\times$

$$\langle \mathbf{H}\phi|\lambda\rangle = \langle\phi|\mathbf{H}^\times\lambda\rangle = \lambda\langle\phi|\lambda\rangle, \quad \forall\phi, \lambda \in \Lambda$$

where Λ is the spectrum of $\mathbf{H}$, $|\lambda\rangle \in \Phi^\times$ and $\mathbf{H}^\times$ is the extension of $\mathbf{H}$ to an operator on the space $\Phi^\times$ defined by

$$\langle \mathbf{H}\phi|\chi\rangle = \langle\phi|\mathbf{H}^\times\chi\rangle, \quad \forall\phi \in \Phi, \quad \forall\chi \in \Phi^\times$$

The completeness of the set of generalized eigenvectors $\{|\lambda\rangle\}_{\lambda\in\Lambda}$ means that for every ϕ, $\psi \in \Phi$ and some uniquely defined positive measure μ on Λ

$$(\phi, \psi)_H = \int_\Lambda d\mu(\lambda)\langle\psi|\lambda\rangle\langle\lambda|\phi\rangle$$

with $\langle\lambda|\phi\rangle = \overline{\langle\phi|\lambda\rangle}$. Thus, we can formally write

$$|\phi\rangle = \int_\Lambda d\mu(\lambda)|\lambda\rangle\langle\lambda|\phi\rangle$$

However, the RHS model for resonances formulated by Bohm and Gadella forms a *nontight rigging* for the Hamiltonian $\mathbf{H}$ in a way that enables the identification of certain elements in the space $\Phi^\times$ as states corresponding to resonances of the quantum mechanical system. These states evolve, for times $t \geq 0$, according to a one-parameter semigroup evolution law generated by $\mathbf{H}^\times$, the extension of $\mathbf{H}$ to $\Phi^\times$. The eigenvectors of $H^\times$ corresponding to resonances have complex eigenvalues and resonances are seen to undergo an exponential decay law for $t \geq 0$.

Despite the fact that the RHS framework allows the possibility of an association of a state with a quantum mechanical resonance, it cannot be said that it provides a complete satisfactory description of quantum mechanical resonances. The main difficulty with such states is that they are elements of $\Phi^\times$, which is not a

Hilbert space. Since $\Phi^\times$ is a space of functionals on Φ, one can evaluate an element $\chi \in \Phi^\times$ on an element $\phi \in \Phi$, i.e., evaluate $\langle\phi|\chi\rangle$. However, an inner product of elements in $\Phi^\times$ is not defined. If a physical observable is associated with a self-adjoint operator, say $\mathbf{A}$, on H, then it is possible to define the extension $\mathbf{A}^\times$ of $\mathbf{A}$ to an operator on $\Phi^\times$, but one cannot extract physical information related to the physical observable since it is impossible to calculate expectation values of $\mathbf{A}^\times$ on elements of $\Phi^\times$. The probabilistic interpretation the quantum mechanical states and the calculation of expectation values corresponding to measurable physical quantities hold in the quantum mechanical Hilbert space framework and do not carry over to $\Phi^\times$.

As mentioned above, in the mathematical framework of the quantum Lax–Phillips scattering theory resonances are associated with Hilbert space state vectors. In this case no problems arise with the calclation of expectation values of observables in the resonant state or with the quantum mechanical probabilistic interpretation. However, the mathematical structure of the Lax–Phillips scattering theory is not directly applicable to standard nonrelativistic quantum mechanics. Originally devised for handling scattering problems within the theory of classical hyperbolic wave equations, the generator of evolution, which we denote by $\mathbf{K}$, in the quantum Lax–Phillips scattering theory, is required to be unbounded from below with a spectrum $\sigma_{ac}(\mathbf{K}) = R$.

In the next few subsections I will describe in more detail the RHS construction in quantum mechanics. In the course of this development I will identify some of the mathematical objects, which are central to the RHS description of resonance phenomena, as identical to those which are found in the Lax–Phillips scattering theory. These relations between the two, seemingly unrelated, theories suggest that a search for a unifying framework, which incorporates the advantages inherent in the two approaches, might prove fruitful.

4.2. RHS for the Free Hamiltonian $\mathbf{H_0}$

Consider a quantum mechanical scattering system exhibiting resonance phenomena. We assume that the model for the system under consideration has the following properties (Bohm and Gadella, 1989).

1. The resonance scattering process is described by a decomposable Hamiltonian

$$\mathbf{H} = \mathbf{H_0} + \mathbf{V}$$

   where $\mathbf{V}$ is the potential term and $\mathbf{H_0}$ is a Hamiltonian describing a free particle.
2. a) The absolutely continuous spectrum of the Hamiltonian $\mathbf{H}$ is $\sigma_{ac}(\mathbf{H}) = R^+$.

b) **H** may have discrete eigenvalues.
c) **H** does not have a singular continuous spectrum $\sigma_{\rm sc}(\mathbf{H}) = 0$.

$$H = H_{\rm p} \oplus H_{\rm ac}$$

where $H_{\rm p} \subset H$ is spanned by the eigenvectors corresponding to the discrete eigenvalues and $H_{\rm ac}$, corresponding to the absolutely continuous spectrum, is its orthogonal complement.

3. The Möller wave operators exist and asymptotic completeness holds.

In addition to the requirements above we simplify our analysis by considering a purely spectral model. This choice is made for the sake of simplicity and clarity of exposition and implies no restriction on the range of applicability of the theory for more general cases.

The first step in the RHS formulation of the scattering process is the construction of two particular RHS, or Gel'fand triplets, $\Phi_\pm \subset H \subset \Phi_\pm^\times$. We choose these RHS, out of the many possible triplets $\Phi \subset H \subset \Phi^\times$, because of their particular suitability for the description of resonances. The spaces $\Phi_\pm$ are chosen such that

1. $\mathbf{H_0}\Phi_\pm \subset \Phi_\pm$;
2. $\mathbf{H_0}$ is e.s.a. on $\Phi_\pm$ (i.e., $\mathbf{H_0}$ with the domain $\Phi_\pm$ is e.s.a. on H); and
3. $\mathbf{H_0}$ is a continuous operator on $\Phi_\pm$ (with respect to the nuclear topology τ_Φ on $\Phi_\pm$).

The spaces of functionals $\Phi_\pm^\times$ are the duals of $\Phi_\pm$, consisting of continuous antilinear functionals respectively on $\Phi_\pm$ (with respect to the topology τ_Φ on $\Phi_\pm$).

A realization of the Gel'fand triple structure in terms of spaces of functions is achieved by using the spectral representation of $\mathbf{H_0}$. We assumed that the continuous spectrum of $\mathbf{H_0}$ (denoted naturally by E) is $\sigma(\mathbf{H_0}) = R^+$. In this case we construct, using the Gel'fand–Maurin theorem, a unitary map $\mathbf{U}$ such that

$$\mathbf{U} : H_{\rm ac} \to L^2(R^+) \tag{4.1}$$

In this spectral representation the free Hamiltonian $\mathbf{H_0}$ is represented by

$$\hat{\mathbf{H}}_\mathbf{0} \equiv \mathbf{U}\mathbf{H_0}\mathbf{U}^{-1} \tag{4.2}$$

The operator $\hat{\mathbf{H}}_\mathbf{0}$ acts as multiplication by the independent variable

$$(\hat{\mathbf{H}}_\mathbf{0}\phi)(E) = E\phi(E), \qquad E \in R^+$$

The spaces $\Phi_\pm$ are then realized as spaces of functions $D_\pm$ given by

$$D_\mp = \mathbf{U}\Phi_\pm \tag{4.3}$$

The unitarity of $\mathbf{U}$ implies that properties 1–3 above are transformed into the following properties of $\hat{\mathbf{H}}_\mathbf{0}$ as an operator on $L^2(R^+)$:

1′. $\hat{\mathbf{H}}_\mathbf{0} D_\pm \subset D_\pm$.

2′. $\hat{\mathbf{H}}_\mathbf{0}$ is e.s.a. on $D_\pm$.
3′. $\hat{\mathbf{H}}_\mathbf{0}$ is continuous on $D_\pm$ (with respect to the nuclear topology on $D_\pm$; see below).

One can find RHS $D_\pm \subset L^2(R^+) \subset D_\pm^\times$ such that $\hat{\mathbf{H}}_\mathbf{0}$ has properties 1′–3′ on $D_\pm$ respectively.

We turn now to the definition of the function spaces $D_\pm$. We define D_+ as a linear space of functions satisfying the following conditions (Bohm and Gadella, 1989):

a. Any $\phi \in D_+$ is the restriction to R^+ of a function in $H^+(R)$ (where $H^+(R)$ consists of boundary values of functions in $H^2(\Pi)$; see section 2).
b. We have $\hat{\mathbf{H}}_\mathbf{0} D_+ \subset D_+$ (where $(\hat{\mathbf{H}}_\mathbf{0}\phi)(E) = E\phi(E)$ for any $\phi \in L^2(R^+)$)
c. D_+ is dense in the Hilbert space $L^2(R^+)$.
d. $\hat{\mathbf{H}}_\mathbf{0}$ is e.s.a. on D_+.
e. We endow D_+ with a complete nuclear, metrizable topology such that $\hat{\mathbf{H}}_\mathbf{0}$ is continuous on D_+.
f. This topology on D_+ is stronger than the Hilbert space topology which D_+ possesses as a subspace of $L^2(R^+)$.

Under these conditions the triplet $D_+ \subset L^2(R^+) \subset D_+^\times$ is a RHS and, defining $\Phi_- \equiv \mathbf{U}^{-1} D_+$, we have that $\Phi_- \subset H \subset \Phi_-^\times$ is also a RHS. Moreover, under conditions a.–f. it is possible to show that, on Φ_-, $\mathbf{H}_\mathbf{0}$ has the properties 1–3 listed above.

A function space D_+ satisfying conditions a–f may be constructed in several steps. Denoting the Schwartz space of functions on R by S, we consider

$$S(R^-) = \{f \mid f \in S \text{ and } \mathrm{Supp}(f) = R^-\} \tag{4.4}$$

We now define another function space,

$$\Delta_+ \equiv \mathbf{F}[S(R^-)] \tag{4.5}$$

where $\mathbf{F}$ is the Fourier transform operator. The Paley–Wiener theorem states that for a function $f \in L^2(R^-)$ we have $\mathbf{F}[f] \in H^+(R)$. For the space Δ_+ we then find that

$$\Delta_+ = \mathbf{F}[S(R^-)] = S \cap H^+(R) \tag{4.6}$$

An important property of the space Δ_+ is the fact that the triplet

$$\Delta_+ \subset H^+(R) \subset \Delta_+^x \tag{4.7}$$

is an RHS.

The space D_+, possessing the required properties a–f, has a simple definition in terms of the space Δ_+

$$D_+ \equiv \mathbf{P}_{\mathbf{R}^+} \Delta_+ \tag{4.8}$$

where $\mathbf{P}_{\mathbf{R}^+}$ is the projection of $L^2(R)$ on $L^2(R^+)$. A theorem of Van Winter (1971) states that any function in $H^+(R)$ is completely determined by its values on R^+. This implies the existence of a map θ such that

$$\theta\Delta_+ = D_+, \qquad \theta^{-1}D_+ = \Delta_+ \tag{4.9}$$

The function space D_+ defined in (4.8) provides us with the desired RHS $D_+ \subset L^2(R^+) \subset D_+^\times$ and, via the map $\mathbf{U}$, the triplet $\Phi_- \subset H \subset \Phi_-^\times$.

As mentioned above, the description of resonance scattering processes requires the definition of two RHS. One of these is defined above, we now define the second. For this we need the function space D_-. The space D_- is defined as a linear space of functions satisfying a set of conditions obtained by replacing D_+ with D_- in conditions b–f above and replacing condition a by

a′. Every function in D_- is the restriction to R^+ of a function in $H^-(R)$.

The construction of the function space D_- is similar to that of D_+. One starts with

$$S(R^+) = \{g | g \in S \text{ and } \mathrm{Supp}\,(g) = R^+\} \tag{4.10}$$

The next step is to define the space Δ_-

$$\Delta_- \equiv \mathbf{F}[S(R^+)] = S \cap H^-(R) \tag{4.11}$$

Again, it is important to note that the triplet

$$\Delta_- \subset H^-(R) \subset \Delta_-^\times \tag{4.12}$$

is an RHS. The definition of D_- in terms of Δ_- is given by

$$D_- \equiv \mathbf{P}_{\mathbf{R}+}\Delta_- \tag{4.13}$$

Using Van Winter's theorem, we can define a one-to-one map $\overline{\theta}$ such that

$$\overline{\theta}\Delta_- = D_- \qquad (\overline{\theta})^{-1}D_- = \Delta_-$$

As in the case of the space D_+ we find that $D_- \subset L^2(R^+) \subset D_-^\times$ is an RHS. Using Eqs. (4.1) and (4.3), we then find the desired RHS $\Phi_+ \subset H \subset \Phi_+^\times$.

4.3. Extension of the Free Hamiltonian and Its Complex Eigenvalues

We have described the construction of the two Gel'fand triples $\Phi_\pm \subset H \subset \Phi_\pm^\times$ as well as their representations, through the mapping $\mathbf{U}$, in terms of the function spaces $D_\pm \subset L^2(R^+) \subset D_\pm^\times$. The RHS structure enables us to extend operators defined on $\Phi_\pm$ to the dual spaces $\Phi_\pm^\times$.

We define the extension to $\Phi_\pm^\times$ of the unperturbed Hamiltonian $\mathbf{H_0}$. This is done using the defining relation

$$\langle \mathbf{H_0}\phi_\pm | f_\pm\rangle = \langle \phi_\pm | (\mathbf{H_0})^\times f_\pm\rangle \qquad \phi_\pm \in \Phi_\pm, \quad f_\pm \in \Phi_\pm^\times \tag{4.14}$$

One of the main results of the RHS approach to resonance scattering is concerned with the existence of complex eigenvalues for the extension of the Hamiltonian $(\mathbf{H_0})^\times$. In fact, if ω_- is complex and $\operatorname{Im}\omega_- \le 0$ then there exists a unique functional $|\omega_-\rangle \in \Phi_+^\times$ such that

$$(\mathbf{H_0})^\times|\omega_-\rangle = \omega_-|\omega_-\rangle \tag{4.15}$$

Similarly, if $\operatorname{Im}\omega_+ \ge 0$, there exists a unique functional $|\omega_+\rangle \in \Phi_-^\times$ such that

$$(\mathbf{H_0})^\times|\omega_+\rangle = \omega_+|\omega_+\rangle \tag{4.16}$$

For each $g \in \Phi_-$ Eq. (4.3) implies that $\tilde{g} = \mathbf{U}g \in D_+ \subset L^2(R^+)$ and, using Eq. (4.9), $\theta^{-1}\tilde{g} = g' \in \Delta_+ \subset H^+(R)$. The Hardy RHS structure of Eq. (4.7) is used to define continuous linear functionals on D_+. This is achieved by the following definition of an extension $\theta^\times$ of the map θ of Eq. (4.9):

$$\langle g'|f\rangle = \langle\theta g'|\theta^\times f\rangle = \langle\tilde{g}|\theta^\times f\rangle, g' \in \Delta_+, \tilde{g} \in D_+,\ f \in \Delta^\times, \theta^\times f \in D_+^\times \tag{4.17}$$

Take the function $f_{\omega_-} \in H^+(R)$ given by $f_{\omega_-}(E) = -(2\pi i)^{-1}(E - \omega_-)^{-1}$, $E \in R$ $\operatorname{Im}\omega_- < 0$. The RHS structure of Eq. (4.7) implies that we can consider f_{ω_-} as an element of $\Delta_+^\times$. In this case we may apply the map $\theta^\times$ through Eq. (4.17),

$$\begin{aligned}\langle g|f_{\omega_-}^\times\rangle &= \langle g|\theta^\times f_{\omega_-}\rangle = \langle\theta^{-1}g|f_{\omega_-}\rangle = (\theta^{-1}g,\ f_{\omega_-})_{H^+(R)}\\ &= -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} dE[\overline{(\theta^{-1}g})(E)]\frac{1}{E-\omega_-} = [\overline{(\theta^{-1}g})(\omega_-)]\end{aligned} \tag{4.18}$$

Using the stability property of D_+ under the action of $\hat{\mathbf{H}}_\mathbf{0}$ (see condition b above), we obtain

$$(\theta^{-1}[\hat{\mathbf{H}}_\mathbf{0}g])(E) = E(\theta^{-1}g)(E), \qquad E \in R \tag{4.19}$$

With the use of Eqs. (4.19), (4.18), and (4.14), we get

$$\begin{aligned}\langle g|(\hat{\mathbf{H}}_\mathbf{0})^\times f_{\omega_-}^\times\rangle &= \langle\hat{\mathbf{H}}_\mathbf{0}g|\theta^\times f_{\omega_-}\rangle = \langle\theta^{-1}[\hat{\mathbf{H}}_\mathbf{0}g]|f_{\omega_-}\rangle\\ &= -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} dE\ E[\overline{(\theta^{-1}g})(E)]\frac{1}{E-\omega_-} = \omega_-[\overline{(\theta^{-1}g})(\omega_-)]\end{aligned} \tag{4.20}$$

Equation (4.18) together with Eq. (4.20) gives

$$\langle g|(\hat{\mathbf{H}}_\mathbf{0})^\times f_{\omega_-}^\times\rangle = \omega_-\langle g|f_{\omega_-}^\times\rangle \tag{4.21}$$

Since $g \in D_+$ is arbitrary, we have

$$(\hat{\mathbf{H}}_\mathbf{0})^\times|f_{\omega_-}^\times\rangle = \omega_-|f_{\omega_-}^\times\rangle \tag{4.22}$$

Define the extension $\mathbf{U}^\times$ of the map $\mathbf{U}$ to $\Phi_+^\times$ through

$$\langle \mathbf{U}g|\mathbf{U}^\times F\rangle = \langle g|F\rangle, \qquad g\in\Phi_+, \quad F\in\Phi_+^\times$$

then, from Eqs. (4.22) and (4.15), we see that $\mathbf{U}^\times|\omega_-\rangle = |f_{\omega_-}^\times\rangle$ hence, we have the desired generalized eigenvector of the extended free Hamiltonian.

4.4. The Evolution Semigroup in the RHS and in Lax–Phillips Theory

In this subsection our goal is to extract further information concerning the semigroup behavior emerging within the RHS formalism via of the action of $\mathbf{H}_\mathbf{0}^\times$ on the generalized eigenvectors $|\omega_-\rangle$ (we note again that there exists a unique generalized eigenvector for each complex ω_- with Im $\omega_- < 0$). We compare this semigroup to the semigroup of the Lax–Phillips theory and show that, in the mathematical sense, both theories utilize the same translation semigroup.

We start with a statement which can easily be proved (Bohm and Gadella, 1989), regarding the stability of $\Phi_\pm$ under the action of elements belonging to the free evolution group $\mathbf{U_0}(t) = e^{-i\mathbf{H_0}t}$

$$\begin{aligned} e^{-i\mathbf{H_0}t}\Phi_- &\subset \Phi_-, && t\le 0\\ e^{-i\mathbf{H_0}t}\Phi_- &\not\subset \Phi_-, && t\ge 0\\ e^{-i\mathbf{H_0}t}\Phi_+ &\subset \Phi_+, && t\ge 0\\ e^{-i\mathbf{H_0}t}\Phi_+ &\not\subset \Phi_+, && t\le 0 \end{aligned} \tag{4.23}$$

The stability properties of Φ_- under the action of $\mathbf{U_0}(t)$ for $t\le 0$ enable us to put forward the following definition of a semigroup evolution on $\Phi_-^\times$

$$\langle e^{i\hat{\mathbf{H}}_\mathbf{0}t}g|k\rangle = \left\langle g\middle|e^{-i(\hat{\mathbf{H}}_\mathbf{0})^\times t}k\right\rangle, \quad t\ge 0,\ g\in D_+,\ k\in D_+^\times \tag{4.24}$$

We now apply the definition (Eq. (4.17)) and obtain, in a similar fashion to Eqs. (4.18)–(4.20),

$$\begin{aligned}\langle g|e^{-i(\hat{\mathbf{H}}_\mathbf{0})^\times t} f_{\omega_-}^\times\rangle &= \langle e^{i\hat{\mathbf{H}}_\mathbf{0}t}g|f_{\omega_-}^\times\rangle = \langle\theta^{-1}[e^{i\hat{\mathbf{H}}_\mathbf{0}t}g]|f_{\omega_-}\rangle\\ &= (\theta^{-1}e^{i\hat{\mathbf{H}}_\mathbf{0}t}g|f_{\omega_-})_{H^+(R)} = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} dE\, e^{-iEt}[(\overline{\theta^{-1}g})(E)]\frac{1}{E-\omega_-}\\ &= e^{-i\omega_- t}[(\overline{\theta^{-1}g})(\omega_-)]\end{aligned} \tag{4.25}$$

Inserting Eq. (4.18) into Eq. (4.25), we obtain

$$\left\langle g\middle|e^{-i(\hat{\mathbf{H}}_\mathbf{0})^\times t} f_{\omega_-}^\times\right\rangle = e^{-i\omega_- t}\langle g|f_{\omega_-}^\times\rangle, \qquad t\ge 0 \tag{4.26}$$

with the following immediate implication

$$e^{-i(\hat{\mathbf{H}}_\mathbf{0})^\times t}|f_{\omega_-}^\times\rangle = e^{-i\omega_- t}|f_{\omega_-}^\times\rangle, \qquad t\ge 0 \tag{4.27}$$

Equation (4.27) exhibits a distinct contractive semigroup behavior. The basic mechanism through which the RHS method achieves this result is clearly demonstrated. Utilizing the Gel'fand triple structure to extend the Hamiltonian (the generator of evolution) to the space of functionals $\Phi^\times$, one may find generalized eigenstates of the extended Hamiltonian, providing the sought for exponential decay law.

To understand better the source of the semigroup law of evolution in Eq. (4.27), we shall make use of the fact that $H^+(R)$ is a closed subspace of $L^2(R)$ and

$$L^2(R) = H^-(R) \oplus H^+(R) \tag{4.28}$$

In Eqs. (4.24)–(4.26) the function g is taken to be an element of D_+, and we have $\theta^{-1}g \in \Delta_+ \subset H^+(R)$. Regarding $H^+(R)$ as a subspace of $L^2(R)$ and denoting by P_+ the orthogonal projection of $L^2(R)$ on $H^+(R)$, we can write $\theta^{-1}g = P_+\theta^{-1}g$. Furthermore, we define a unitary multiplicative operator $\widehat{e^{-iEt}} : L^2(R) \to L^2(R)$

$$(\widehat{e^{-iEt}} f)(E) = e^{-iEt} f(E), \qquad f \in L^2(R) \tag{4.29}$$

Inserting this information in Eq. (4.25) and taking notice of the fact that the scalar product in $H^+(R)$ is inherited from that of $L^2(R)$, we get

$$\begin{aligned} \langle g|e^{-i(\hat{\mathbf{H}}_0)^\times t} f^\times_{\omega_-}\rangle &= -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} dE\,[(\overline{\theta^{-1}g})(E)]\,e^{-iEt}\frac{1}{E-\omega_-} \\ &= (\theta^{-1}g, \widehat{e^{-iEt}} f_{\omega_-})_{L^2(R)} = (P_+\theta^{-1}g, \widehat{e^{-iEt}} f_{\omega_-})_{L^2(R)} \\ &= (\theta^{-1}g, P_+\widehat{e^{-iEt}} f_{\omega_-})_{L^2(R)} = (\theta^{-1}g, T_{e^{-iEt}} f_{\omega_-})_{H^+(R)} \end{aligned} \tag{4.30}$$

where $T_{e^{-iEt}}$ is the Toeplitz operator on $H^+(R)$ with symbol e^{-iEt} defined by (see Eq. (3.8) for the vactor-valued case)

$$T_{e^{-iEt}} f = P_+\widehat{e^{-iEt}} f, \qquad f \in H^+(R) \tag{4.31}$$

On the other hand we have, from Eqs. (4.18) and (4.26),

$$\langle g\big|e^{-i(\hat{\mathbf{H}}_0)^\times t} f^\times_{\omega_-}\rangle = e^{-i\omega_- t}\langle g|\, f^\times_{\omega_-}\rangle = e^{-i\omega_- t}(\theta^{-1}g,\ f_{\omega_-})_{L^2(R)}, \quad t \geq 0 \tag{4.32}$$

Comparing Eqs. (4.30) and (4.32), we find

$$(\theta^{-1}g, T_{e^{-iEt}} f_{\omega_-})_{H^+(R)} = e^{-i\omega_- t}(\theta^{-1}g,\ f_{\omega_-})_{H^+(R)} \tag{4.33}$$

where $\theta^{-1}g \in \Delta_+$. It is easy to check that in Eq. (4.33) it is possible to replace the state $\theta^{-1}g$ by any element of $H^+(R)$, hence we find that

$$T_{e^{-iEt}} f_{\omega_-} = e^{-i\omega_- t} f_{\omega_-}, \qquad t \geq 0, \quad f_{\omega_-} \in H^+(R) \tag{4.34}$$

Equation (4.34) implies that the application of the extended evolution $e^{-i(\hat{\mathbf{H}}_0)^{\times}t}$ to the generalized eigenstate $|f^{\times}_{\omega_-}\rangle$ correspond to the application, in $H^+(R)$, of the Toeplitz operator $T_{e^{-iEt}}$ to its eigenfunction $f_{\omega_-} \in H^+(R)$.

Comparison of Eqs. (4.30)–(4.34) with the structure described in section 3, and especially Eqs. (4.30) and (3.8), demonstrates clearly that both in the RHS formalism and in the Lax–Phillips theory the free evolution is associated with the same translation semigroup of Toeplitz operators. Some of the eigenvectors of this semigroup are identified eventually as corresponding to the decaying resonant states. In the case of the Lax–Phillips theory the mechanism for this identification is described in section 3. The corresponding mechanism for the RHS model is discussed below. A comparison between the two reveals surprising relations.

4.5. Extension of the Full Hamiltonian and Its Resonance Eigenstates

We turn now to a discussion of the resonance states of the full, interacting, Hamiltonian **H**. One defines two RHS, as in the case of the free Hamiltonian $\mathbf{H_0}$, suitable for the definition of the extension $\mathbf{H}^{\times}$ of the full Hamiltonian. To define these spaces, an extension of the Möller wave operators is needed. The basic assumptions at the beginning of subsection 4.2 include the existence and asymptotic completeness of the Möller wave operators. Asymptotic completeness implies that

$$H_{\text{ac}} = \mathbf{\Omega}^{\pm} H \tag{4.35}$$

where $\mathbf{\Omega}^{\pm}$ are the Möller wave operators and H_{ac} is as in subsection 4.2. We define

$$\Phi^+ \equiv \mathbf{\Omega}^+\Phi_+, \qquad \Phi^- \equiv \mathbf{\Omega}^-\Phi_- \tag{4.36}$$

where $\Phi_\pm$ are the "smaller" spaces of the Gel'fand triplets $\Phi_\pm \subset H \subset \Phi^{\times}_{\pm}$ defined in subsection 4.2 (see, for example, Eq. (4.3)). One can show (Bohm and Gadella, 1989) that

$$\Phi^+ \subset H_{\text{ac}} \subset (\Phi^+)^{\times}, \quad \Phi^- \subset H_{\text{ac}} \subset (\Phi^-)^{\times} \tag{4.37}$$

are RHS. The extension of the Möller wave operators is achieved via the following definitions

$$\langle \phi \text{I} F_+\rangle = \langle \mathbf{\Omega}^+\phi|(\mathbf{\Omega}^+)^{\times} F_+\rangle, \qquad \forall \phi \in \Phi_+, \quad F_+ \in (\Phi_+)^{\times} \tag{4.38}$$

$$\langle \phi \text{I} F_-\rangle = \langle \mathbf{\Omega}^-\phi|(\mathbf{\Omega}^-)^{\times} F_-\rangle, \qquad \forall \phi \in \Phi_-, \quad F_- \in (\Phi_-)^{\times} \tag{4.39}$$

Using the definitions (Eq. (4.38) and (4.39)), one has immediately

$$(\mathbf{\Omega}^+)^{\times} F_+ \in (\Phi^+)^{\times}, (\mathbf{\Omega}^-)^{\times} F_- \in (\Phi^-)^{\times}, F_- \in \Phi^{\times}_+, F_- \in \Phi^{\times}_- \tag{4.40}$$

In fact, it is possible to show further that

$$(\mathbf{\Omega}^+)^{\times}(\Phi_+)^{\times} = (\Phi^+)^{\times}, \quad (\mathbf{\Omega}^+)^{\times}(\Phi_-)^{\times} = (\Phi^-)^{\times} \tag{4.41}$$

Denote the restriction of the full Hamiltonian $\mathbf{H}$ to $H_{\rm ac}$ by $\mathbf{H}'$. Then $\mathbf{H}'$ has the same properties on $\Phi^\pm$ as $\mathbf{H_0}$ on $\Phi_\pm$, i.e.,

1″. $\mathbf{H}'\Phi^\pm \subset \Phi^\pm$.
2″. $\mathbf{H}'$ is e.s.a. on $\Phi^\pm$.
3″. $\mathbf{H}'$ is a continuous opertor on $\Phi^\pm$.

Then, by the same procedure that was applied above to $\mathbf{H_0}$ (see Eq. (4.14)) it is possible to define the extension of $\mathbf{H}'$ to $(\Phi^\pm)^\times$. The defining relation is

$$\langle \mathbf{H}'\phi | F\rangle = \langle \phi | (\mathbf{H}')^\times F\rangle, \qquad \phi \in \Phi^\pm, \quad F \in (\Phi^\pm)^\times \tag{4.42}$$

We have shown above the existence of a functional $|\omega_-\rangle$ satisfying Eq. (4.15) (existence of the functional $|\omega_+\rangle$ can be shown in a similar manner). Following Eqs. (4.38)–(4.41), we denote

$$|\omega^\pm\rangle = (\mathbf{\Omega}^\pm)^\times |\omega_\pm\rangle \tag{4.43}$$

We have then

$$\begin{aligned}\langle g|(\mathbf{H}')^\times \omega^\pm\rangle &= \langle g|(\mathbf{H}')^\times (\mathbf{\Omega}^\pm)^\times \omega_\pm\rangle = \langle (\mathbf{\Omega}^\pm)^{-1}\mathbf{H}' g|\omega_\pm\rangle \\ &= \langle \mathbf{H_0}(\mathbf{\Omega}^\pm)^{-1} g|\omega_\pm\rangle = \langle (\mathbf{\Omega}^\pm)^{-1} g|\mathbf{H_0}^\times \omega_\pm\rangle = \omega_\pm \langle (\mathbf{\Omega}^\pm)^{-1} g|\omega_\pm\rangle \\ &= \omega_\pm \langle g|(\mathbf{\Omega}^\pm)^\times \omega_\pm\rangle = \omega_\pm \langle g|\omega^\pm\rangle \end{aligned} \tag{4.44}$$

for $g \in \Phi^\pm$ and, hence,

$$(\mathbf{H}')^\times |\omega^\pm\rangle = \omega_\pm |\omega^\pm\rangle \tag{4.45}$$

As in the case of the free evolution $(\mathbf{H}')^\times$ generates for $t \geq 0$ an evolution semigroup

$$\left\langle g \middle| e^{-i(\mathbf{H}')^\times t}\omega^+\right\rangle = e^{-i\omega_+ t}\langle g|\omega^+\rangle, \qquad g \in \Phi^+, \quad t \geq 0 \tag{4.46}$$

Consider now the representation of Eq. (4.46) in terms of the RHS $D_+ \subset L^2(R^+) \subset D_+^\times$. Thus, we apply the map $\mathbf{U}$ (and its extension $\mathbf{U}^\times$) and find, using Eq. (4.44) and Eq. (4.30)

$$\begin{aligned}\left\langle g \middle| e^{-i(\mathbf{H}')^\times t}\omega^+\right\rangle &= \langle e^{i\mathbf{H_0}t}(\mathbf{\Omega}^+)^{-1} g|\omega_+\rangle = \langle \mathbf{U}\, e^{i\mathbf{H_0}t}(\mathbf{\Omega}^+)^{-1} g|\mathbf{U}\omega_+\rangle \\ &= \langle e^{i\hat{\mathbf{H}}_0 t}\mathbf{U}(\mathbf{\Omega}^+)^{-1} g|\, f_{\omega_+}^\times\rangle = \langle \mathbf{U}\,(\mathbf{\Omega}^+)^{-1} g|\, e^{-i(\hat{\mathbf{H}}_0)^\times t}\, f_{\omega_+}^\times\rangle \\ &= (\theta^{-1}\mathbf{U}(\mathbf{\Omega}^+)^{-1} g,\, T_{e^{-iEt}}\, f_{\omega_+})_{H^+(R)}\end{aligned} \tag{4.47}$$

Furthermore, we have

$$\begin{aligned}\langle g|\omega^+\rangle &= \langle (\mathbf{\Omega}^+)^{-1} g|\omega_+\rangle = \langle \mathbf{U}(\mathbf{\Omega}^+)^{-1} g|\mathbf{U}\omega_+\rangle \\ &= \langle \mathbf{U}(\mathbf{\Omega}^+)^{-1} g|\, f_{\omega_+}^\times\rangle = (\theta^{-1}\mathbf{U}(\mathbf{\Omega}^+)^{-1} g,\ f_{\omega_+})_{H^+(R)}\end{aligned} \tag{4.48}$$

and we see that the application of the extended (full) evolution $e^{-i(\mathbf{H}')^{\times}t}$, $t \geq 0$ to the generalized eigenstate $|\omega^+\rangle$ correspond to the application of the Toeplitz opertor $T_{e^{-iEt}}$ on $H^+(R)$ to the eigenfunction $f_{\omega_+} \in H^+(R)$.

After the appropriate machinery has been developed in this and previous subsections, it is possible to introduce the RHS model for resonances. This is done in the next subsection. The relation of the mathematical structure of the RHS model of resonances to that of the Lax–Phillips scattering theory is then made explicit.

4.6. The RHS Model of Resonances and the Mathematical Structure of the Lax–Phillips Scattering Theory

Let us first complete the RHS model for resonances. The existence and asymptotic completeness of the Möller wave operators guarantees the existence and unitarity of the scattering operator $\mathbf{S} = (\mathbf{\Omega}^-)^{\dagger}\mathbf{\Omega}^+$. Consider a generic matrix element of $\mathbf{S}$

$$\begin{aligned}(\phi^{\text{out}}, \mathbf{S}\psi^{\text{in}})_H &= (\phi^{\text{out}}, (\mathbf{\Omega}^-)^{\dagger}\mathbf{\Omega}^+\psi^{\text{in}})_H \\ &= (\mathbf{\Omega}^-\phi^{\text{out}}, \mathbf{\Omega}^+\psi^{\text{in}})_H, \qquad \phi^{\text{out}}, \psi^{\text{in}} \in H \end{aligned} \tag{4.49}$$

To make use of the RHS $\Phi_{\pm} \subset H \subset \Phi_{\pm}^{\times}$, the definition Eq. (4.36), and the RHS of Eq. (4.37), we take $\phi^{\text{out}} \in \Phi_-$ and $\psi^{\text{in}} \in \Phi_+$ (then, according to Eq. (4.36), we have $(\mathbf{\Omega}^-\phi) \in \Phi^-$, $(\mathbf{\Omega}^+\psi) \in \Phi^+$).

We can now use the unitary map $\mathbf{U}$ of Eq. (4.1) to write the matrix element of Eq. (4.49) in the form

$$(\phi^{\text{out}}, \psi^{\text{in}})_H = (\mathbf{U}\phi^{\text{out}}, \mathbf{U}\mathbf{S}\mathbf{U}^{-1}\mathbf{U}\psi^{\text{in}})_{L^2(R^+)} \tag{4.50}$$

since $\phi^{\text{out}} \in \Phi_-$ we have $\mathbf{U}\phi^{\text{out}} \in D_+$ and, similarly, $\mathbf{U}\psi^{\text{in}} \in D_-$. According to the nuclear spectral (Gel'fand–Maurin) theorem we can use the corresponding complete sets of generalized eigenvectors and write

$$(\mathbf{U}\phi^{\text{out}})(E) = \langle E_-|\phi^{\text{out}}\rangle, \qquad (\mathbf{U}\psi^{\text{in}})(E) = \langle E_+|\psi^{\text{in}}\rangle \tag{4.51}$$

where

$$|f\rangle = \int_0^{\infty} dE|E_-\rangle\langle E_-|f\rangle \qquad f \in D_-$$

and

$$|g\rangle = \int_0^{\infty} dE|E_+\rangle\langle E_+|g\rangle \qquad g \in D_+$$

We note also that, since $\mathbf{U}\phi^{\text{out}} \in D_+$, we have $(\mathbf{U}\phi^{\text{out}})^* \in D_-$. Therefore, both $\langle E_+|\psi^{\text{in}}\rangle$ and $\langle\phi^{\text{out}}|E_-\rangle$ can be extended to Hardy class functions in the lower half-plane or, more precisely, to functions in Δ_+.

Calculating the "energy representation" matrix elements $\mathbf{USU}^{-1}$ in Eq. (4.50), we obtain the well-known "energy representation" $S(E)$ of the S-matrix. For $E \geq 0$ the function $S(E)$ is a boundary value of a function $S(\omega)$ (ω complex) analytic above the cut on R_+. Thus we obtain

$$(\phi^{\text{out}} S\psi^{\text{in}})_H = \int_0^\infty dE\,\langle\phi^{\text{out}}|E_-\rangle\langle E_+|\psi^{\text{in}}\rangle S(E + i\epsilon) \tag{4.52}$$

Making use of the definitions Eqs. (4.38) and (4.39) and Eqs. (4.40) and (4.41) we can write

$$\langle\phi^{\text{out}}|E_-\rangle = \langle\mathbf{\Omega}^-\phi^{\text{out}}|(\mathbf{\Omega}^-)^\times E_-\rangle = \langle\mathbf{\Omega}^-\phi^{\text{out}}|E^-\rangle \tag{4.53}$$

and

$$\langle\psi^{\text{in}}|E_+\rangle = \langle\mathbf{\Omega}^+\psi^{\text{in}}|(\mathbf{\Omega}^+)^\times E_+\rangle = \langle\mathbf{\Omega}^+\psi^{\text{in}}|E^+\rangle \tag{4.54}$$

where $|E^-\rangle = (\mathbf{\Omega}^-)^\times|E_-\rangle$, $|E^+\rangle = (\mathbf{\Omega}^+)^\times|E_+\rangle$. With the help of Eqs. (4.53) and (4.54) we rewrite Eq. (4.52) in the form

$$(\phi^{\text{out}}, S\psi^{\text{in}})_H = \int_0^\infty dE\,\langle\mathbf{\Omega}^-\phi^{\text{out}}|E^-\rangle\langle E^+|\mathbf{\Omega}^+\psi^{\text{in}}\rangle S(E + i\epsilon) \tag{4.55}$$

To complete the RHS model for resonances, we need to assume certain properties satisfied by the analytic continuation of the function $S(E + i\epsilon)$ from above the cut on the positive real axis to the lower half-plane of the second Riemann sheet. We denote the analytically continued function by $S_{II}(\omega)$. Then the assumption is (Bohm and Gadella, 1989) that $S_{II}(\omega)$ is polynomially bounded at infinity, i.e., that there exists some polynomical function $P(\omega)$, $\omega \in C$, such that

$$|S_{II}(\omega)| \leq |P(\omega)|, \qquad |\omega| > R_0 \tag{4.56}$$

for some $R_0 > 0$.

For simplicity we assume that we are dealing with only a single resonance, associated with a simple pole of the S-matrix in the second sheet lower half-plane. Then $S_{II}(\omega)$, $\text{Im}(\omega) < 0$ is meromorphic in the second sheet lower half-plane and is bounded by a polynomial growth as $|\omega| \to \infty$.

Consider the following integral defined in the second Riemann sheet

$$\int_C d\omega\langle\mathbf{\Omega}^-\phi^{\text{out}}|\omega^-\rangle\langle\omega^+|\mathbf{\Omega}^+\psi^{\text{in}}\rangle S_{II}(\omega) \tag{4.57}$$

where C is a small circle in the second Riemann sheet, enclosing the location of the pole of $S_{II}(\omega)$ which we take to be at the point $\omega = Z_{\text{R}}$. We note that according to the assumptions specified above the argument of the integral in Eq. (4.57) is such that the integral is well defined. The residue theorem then implies that we have

$$\int_C d\omega \langle \boldsymbol{\Omega}^- \phi^{\text{out}} | \omega^- \rangle \langle \omega^+ | \boldsymbol{\Omega}^+ \psi^{\text{in}} \rangle S_{II}(\omega)$$
$$= -2\pi i \langle \boldsymbol{\Omega}^- \phi^{\text{out}} | Z_R^- \rangle \langle Z_R^+ | \boldsymbol{\Omega}^+ \psi^{\text{in}} \rangle \text{Res}[(S_{II}(\omega)), Z_R]$$

The same assumptions allow us to deform the circle C into a contour C' consisting of a straight segment in the second sheet, just below the real axis, and an arc, the radius of which we can take to infinity. The polynomial bound on the growth of $S_{II}(\omega)$ then ensures that the integral on the arc does not contribute when its radius is taken to infinity. Thus we obtain

$$\int_{-\infty}^{\infty} dE \langle \boldsymbol{\Omega}^- \phi^{\text{out}} | (E - i\epsilon)^- \rangle \langle (E - i\epsilon)^+ | \boldsymbol{\Omega}^+ \psi^{\text{in}} \rangle S_{II}(E - i\epsilon)$$
$$= -2\pi i \langle \boldsymbol{\Omega}^- \phi^{\text{out}} | Z_R^- \rangle \langle Z_R^+ | \boldsymbol{\Omega}^+ \psi^{\text{in}} \rangle \text{Res}[(S_{II}(\omega)), Z_R] \tag{4.58}$$

The analyticity properties of $S_{II}(\omega)$ imply that we can cross the cut on R^+ and obtain

$$\int_0^{\infty} dE \langle \boldsymbol{\Omega}^- \phi^{\text{out}} | E^- \rangle \langle E^+ | \boldsymbol{\Omega}^+ \psi^{\text{in}} \rangle S(E - i\epsilon)$$
$$= -\int_{-\infty}^{0} dE \langle \boldsymbol{\Omega}^- \phi^{\text{out}} | E^- \rangle \langle E^+ | \boldsymbol{\Omega}^+ \psi^{\text{in}} \rangle S_{II}(E)$$
$$-2\pi i \langle \boldsymbol{\Omega}^- \phi^{\text{out}} | Z_R^- \rangle \langle Z_R^+ | \boldsymbol{\Omega}^+ \psi^{\text{in}} \rangle \text{Res}[(S_{II}(\omega)), Z_R] \tag{4.59}$$

Denote

$$\phi^- = \boldsymbol{\Omega}^- \phi^{\text{out}}, \qquad \psi^+ = \boldsymbol{\Omega}^+ \psi^{\text{in}} \tag{4.60}$$

where ϕ^- and ψ^+ are the interacting states corresponding asymptotically to ϕ^{out} and ψ^{in} for $t \to +\infty$ and $t \to -\infty$ respectively. With Eqs. (4.60) and (4.55) we can write Eq. (4.59) as

$$(\phi^-, \psi^+)_H = (\phi^{\text{out}}, \mathbf{S}\psi^{\text{in}})_H = -\langle \phi^- | \left(\int_{-\infty}^{0} dE | E^- \rangle \langle E^+ | S_{II}(E) \right) | \psi^+ \rangle$$
$$-2\pi i \text{Res}[(S_{II}(\omega)), Z_R] \langle \phi^- | Z_R^- \rangle \langle Z_R^+ | \psi^+ \rangle \tag{4.61}$$

We can then say that

$$-\int_{-\infty}^{0} dE | E^- \rangle \langle E^+ | S_{II}(E) - 2\pi i \text{Res}[(S_{II}(\omega)), Z_R] | Z_R^- \rangle \langle Z_R^+ | \tag{4.62}$$

spans, in the sense of Eqs. (4.59) and (4.60), the space of interacting states. The second term in Eq. (4.62) is identified as the resonant state. We see from Eq. (4.46) that

$$e^{-i(\mathbf{H}')\times t}|Z_R\rangle = e^{-iZ_R t}|Z_R\rangle, \qquad t \geq 0 \tag{4.63}$$

We now cast the development of the RHS model for resonances into the language of the general mathematical context of the Lax–Phillips scattering theory. We start with an analysis of the assumed properties of the S-matrix as given above. We first note that in Eq. (4.52) $S(E + i\epsilon)$ is the boundary value on R^+ of a function $S(\omega)$ analytic in the first sheet upper half-plane above the cut. The unitarity of the S-matrix implies that on R^+ we have $|S(E)| = 1$. The analytic continuation of $S(\omega)$ to the lower half-plane across R^+ is then given by

$$S_{II}(\omega) = (S^*(\bar{\omega}))^{-1}, \qquad \mathrm{Im}\,\omega < 0 \tag{4.64}$$

Denote $\omega' = \bar{\omega}$, $\mathrm{Im}\omega < 0$. Equation (4.64) implies the following:

$$|S_{II}(\omega)| = |(S^*(\bar{\omega}))^{-1}| = |(S^*(\bar{\omega}))|^{-1} = |S^*(\omega')|^{-1} = |S(\omega')|^{-1} \tag{4.65}$$

and so

$$|S(\omega')| = |S_{II}(\omega)|^{-1} \tag{4.66}$$

To proceed at this point, we make one more assumption on the behavior of $S_{II}(\omega)$ in addition to the polynomial growth restriction of the RHS model, Eq. (4.56). Suppose that there is a constant $C > 0$ such that $|S_{II}(\omega)| > C$ for all $\mathrm{Im}\,\omega < 0$. In this case we have, using Eq. (4.66),

$$|S(\omega')| = |S_{II}(\omega)|^{-1} < C, \qquad \omega' = \bar{\omega}, \quad \mathrm{Im}\,\omega' > 0 \tag{4.67}$$

and we find that $S(\omega')$ is a bounded analytic function on the first sheet upper half-plane. The limit of this function on the real axis is then also bounded and this implies that $S(\omega)$ cannot have bound state poles on the negative real axis. We see that the condition Eq. (4.67) is rather strong and limits the class of models which can be considered. However, this assumption enables us to use very powerful tools from the theory of H^p spaces and obtain some surprising results, as we shall see presently. These results, in turn, serve as motivation for the construction of a more general structure. We note that the restricting assumption does allow us to consider simple models of resonance scattering, for example, a simple (Friedrichs, 1950) model in which the pertubed Hamiltonian $\mathbf{H}$ has a continuous spectrum equal to R^+, a resonance pole and no bound states.

Denote the space of all bounded analytic functions on the upper half-plane by $H^\infty(\prod)$. Putting forward the condition in Eq. (4.67), we can write $S \in H^\infty(\prod)$. We then continue by introducing several definitions and theorems concerning H^p spaces of functions on the upper half-plane (the theory of scalar valued H^p

functions can be found, for example, in Duren, 1970; Hoffman, 1962. For the vector-valued case, see Rosenblum and Rovnyak (1985) or Nikol'skii (1985).

Definition (the spaces $H^p(\prod)$). The space of analytic functions on the upper half-plane such that $|f(x+iy)|^p (0 < p < \infty)$ is integrable on x for each $y > 0$ and

$$\sup_{y>0} \left\{ \int_{-\infty}^{+\infty} |f(x+iy)|^p dx \right\}^{1/p} < C$$

for some $C > 0$ is called a $H^p(\prod)$ space $(0 < p < \infty)$. We note that if $|S_{II}(\omega)|$ rises fast enough as $|\omega| \to \infty$ in the lower half-plane (i.e., $|S(\omega)|$ falls fast enough on the upper half-plane) we may have $S(\omega) \in H^p(\prod)$ for some $0 < p < \infty$.

We state several theorems providing the structure of a function $f \in H^p(\prod)$, $p > 0$.

Definition (Blaschke product). A Blaschke product on the upper half-plane is an analytic function $b(z)$ on $\prod$ of the form

$$b(z) = \left(\frac{z-i}{z+i} \right)^m \prod_n \frac{|z_1^2+1|}{z_1^2+1} \cdot \frac{z-z_n}{z-\overline{z}_n} \tag{4.68}$$

Here m is a nonnegative integer and z_n, Im $z_n > 0$, are zeros of $b(z)$ in $\prod$, finite or infinite in number.

Theorem A. *If $f \in H^p(\prod)$ $(0 < p \leq \infty)$ and $f \not\equiv 0$, then $f(z) = b(z)g(z)$, where $g(z)$ is a nonvanishing $H^p(\prod)$ function. The boundary value function on R satisfies $|g(x)| = |f(x)|$ a.e., $x \in R$. The function $b(z)$ is a Blaschke product of the form given in Eq. (4.68) and z_n are the zeros ($z_n \neq i$) of f in $\prod$.*

We define a scalar inner function on Π. This definition is to be compared with the properties of the Lax–Phillips S-matrix, which is a more general, operator-valued inner function.

Definition (inner function). A function f, analytic in the upper half-plane, with the property $|f(z)| < 1$ for $z > 0$ and such that the boundary value function of f on R satisfies $|f(x)| = 1$, $x \in R$ a.e. is called an inner function (for $\prod$).

Definition (singular inner function). A singular inner function (on $\prod$) is a function of the form

$$s(z) = \exp \left\{ i \int_{-\infty}^{+\infty} \frac{1+tz}{t-z} \, d\nu(t) \right\}, \quad z \in \prod \tag{4.69}$$

where $d\nu(t)$ is a singular measure on R.

Theorem B (canonical factorization of an inner function). *Every inner function on* $\prod$, $f_{\text{in}}(z) \not\equiv 0$, *can be factorized in the form*

$$f_{\text{in}}(z) = e^{i\alpha z} b(z) s(z), \qquad z \in \prod \tag{4.70}$$

where $\alpha \geq 0$, $b(z)$ *is a Blaschke product, and* $s(z)$ *is a singular inner function.*

Definition (outer function). An outer function on $\prod$ is a function of the form

$$G(z) = e^{i\gamma} \exp\left\{ -i \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1+tz}{t-z} \cdot \frac{\log \omega(t)}{1+t^2}\, dt \right\} \tag{4.71}$$

for some real γ and some measurable function $\omega(t) \geq 0$ with the properties

$$\text{a. } \int_{-\infty}^{+\infty} \frac{\log \omega(t)}{1+t^2}\, dt > -\infty \quad \text{b. } \int_{-\infty}^{+\infty} \frac{[\omega(t)]^p}{1+t^2}\, dt < \infty \tag{4.72}$$

Theorem C (canonical factorization theorem). *Every function* $f(z)$ *of the class* $H^p(\prod)$, $0 < p < \infty$, *has a unique factorization into an inner and outer functions, i.e., there exists a factorization of* $f(z)$ *of the form*

$$f(z) = f_{\text{in}}(z) f_{\text{out}}(z) = e^{i\alpha z} b(z) s(z) G(z) \tag{4.73}$$

where $b(z)$ *is a Blaschke product,* $s(z)$ *is a singular inner function,* $G(z)$ *is an outer function with the condition (b) of Eq. (4.72) replaced by* $\omega \in L^p$. *Conversely, each product of such factors belongs to* $H^p(\prod)$.

Application of Theorem A above to $S(\omega)$, $\text{Im}\,\omega > 0$ satisfying Eq. (4.67) (i.e., when $S(\omega) \in H^\infty(\prod)$) shows that in this case $S(\omega)$, $\omega \in \prod$ has the factorization

$$S(\omega) = B_S(\omega)\tilde{S}(\omega), \qquad \omega \in \prod \tag{4.74}$$

where $B_S(\omega)$ is a Blaschke product providing all the zeros of $S(\omega)$ in the upper half-plane and $\tilde{S}(\omega)$ is nonvanishing on Π. If, in addition, $|S(\omega)|$ falls fast enough on the upper half-plane, such that $S(\omega) \in H^p(\prod)$ for some $0 < p < \infty$ (i.e., $|S_{II}(\omega)|$ rises fast enough as $|\omega| \to \infty$ in the lower half-plane), then we can apply Theorem C and find

$$S(\omega) = S_{\text{in}}(\omega) S_{\text{out}}(\omega) = e^{i\alpha\omega} B_S(\omega) S_{\text{sing}}(\omega) S_{\text{out}}(\omega), \quad \text{Im}\,\omega > 0 \tag{4.75}$$

where $S_{\text{sing}}(\omega)$ is the singular part of $S_{\text{in}}(\omega)$ and in Eq. (4.74)

$$\tilde{S}(\omega) = e^{i\alpha\omega} S_{\text{sing}}(\omega) S_{\text{out}}(\omega).$$

Let us consider a situation where $B_S(\omega)$ in the canonical factorization of $S(\omega)$ is a simple, single Blaschke factor (this corresponds to a single simple resonance as we shall see below).

$$B_S(\omega) = \frac{|Z_R^2 + 1|}{Z_R^2 + 1} \cdot \frac{\omega - \overline{Z}_R}{\omega - Z_R}, \quad \operatorname{Im} Z_R < 0 \tag{4.76}$$

In this case we find that $S(\omega)$, the analytic continuation of the S-matrix into the upper half-plane, takes the form

$$S(\omega) = e^{i\alpha\omega} \frac{|Z_R^2 + 1|}{Z_R^2 + 1} \cdot \frac{\omega - \overline{Z}_R}{\omega - Z_R} S_{\text{sing}}(\omega) S_{\text{out}}(\omega), \quad \operatorname{Im} \omega > 0 \tag{4.77}$$

and the Blaschke factor provides the only zero of $S(\omega)$ in the upper half-plane (the same factor generates the second sheet pole of the analytic continuation $S_{II}(\omega)$ of $S(\omega)$ to the second Riemann sheet). The RHS model S-matrix is then the boundary value of $S(\omega)$ on R^+ (under the added assumptions on $S_{II}(\omega)$ mentioned above). Conversely, given the RHS model S-matrix, we may extend it to an analytic function on the upper half-plane such that $S(\omega)$ is a well-defined operator $S(\omega) : H^p(\prod) \to H^p(\prod)$.

Finally, to conclude the argument, we need one more theorem from the theory of H^p functions

Theorem D. *Let $f \in H^p (1 \leq p < \infty)$ and $f \not\equiv 0$. Let f_{in} be the inner part in the canonical factorization of f. Then*

$$f H^p = f_{\text{in}} H^p \tag{4.78}$$

According to Theorem D the action of $S(\omega)$, in the form given in Eq. (4.75) or (4.77), as a multiplicative operator on the Hardy space$H^+(R)$ give

$$S H^+(R) = S_{\text{in}} H^+(R) \tag{4.79}$$

where S_{in} is the inner part of S.

We have seen that the action of the extension of the full evolution semigroup $e^{-i(H')^\times t}$, $t \geq 0$, on the generalized eigenstate $|\omega_+\rangle$ correspond to the action of the semigroup of Toeplitz operators $T_{e^{-iEt}}$ on the eigenfunction $f_{\omega_+} \in H^+(R)$. The discussion in sections 2 and 3 (particularly Eqs. (3.8) and (3.9)) shows that the same correspondence exists between the semigroup $\{T(t)\}_{t \geq 0}$ defined on the Lax–Phillips Hilbert space (see Eq. (3.1)) and its representation in the outgoing spectral representation of the Lax–Phillips scattering theory. Equation (4.79) shows that with a given RHS model S-matrix (satisfying certain conditions) we can associate a unique operator on $H^+(R)$, i.e.,

$$S^{\text{RHS}} \to S_{\text{in}}$$

where S^{RHS} is the RHS model S-matrix and S_{in} is its inner part. Furthermore, as in the case of the semigroup, one finds in the Lax–Phillips scattering theory an object corresponding to S_{in}, i.e., the Lax–Phillips S-matrix.

We go on to show that the proof of the main theorem of the Lax–Phillips theory (theorem C of section 2) enables us to identify the eigenvectors of the semigroup generated by the extended evolution in the RHS model. Let $\hat{K} \subset H^+(R)$ be the following subspace:

$$\hat{K} = H^+(R) \ominus SH^+(R) = H^+(R) \ominus S_{\text{in}} H^+(R) \tag{4.80}$$

(compare with Eq. (3.12)) then we can define a Lax–Phillips-type semigroup

$$\hat{Z}(t) = T_{e^{-iEt}} | \hat{K}, \qquad t \geq 0 \tag{4.81}$$

In the case of scalar functions with which we are dealing here the Lax–Phillips theorem reduces to the statement that the eigenvalues of $\hat{Z}(t)$ correspond to zeros of $S_{\text{in}}(\omega)$ in the upper half-plane, or poles of its analytic continuation to the lower half-plane. From Eq. (4.77) we see that the only zero of $S_{\text{in}}(\omega)$ in the upper half-plane, which according to Theorem A in this section is also the only zero of $S(\omega)$, is at $\overline{Z}_R$. The analytic continuation of $S(\omega)$ to the lower half-plane of the second Riemann sheet through Eq. (4.64) (see also Eq. (2.8)) implies that the location of the pole is at Z_R. From the proof of the Lax–Phillips theorem we deduce that the corresponding eigenvector of $\hat{Z}(t)$ is given by

$$f_{Z_R}(E) = \frac{i}{E - Z_R}, \qquad f_{Z_R} \in \hat{K} \tag{4.82}$$

which is exactly the generalized eigenvector of the evolution semigroup generated by $e^{-i(H')^{\times}t}$, $t \geq 0$ in the Hardy RHS.

It should be emphasized at this point that one can generalize the structures and arguments given in this section to the more general case of vector-valued and operator-valued functions, using the theory of vector- and operator-valued H^p functions. Amongst other things the discussion of section 4 gives a small glimpse into the powerful structure of the theory of H^p functions and its potential utilization within the framework of quantum scattering theory.

5. CONCLUSIONS

After an introduction to the Lax–Phillips scattering theory in section 2 some parts of the mathematical structure of this theory were put into a more general mathematical context in section 3. Then, in section 4 the RHS model for quantum mechanical resonances was revisited. It is shown in section 4 that, at least for certain simple quantum mechanical scattering problems, the evolution semigroup of the RHS model, as well as its eigenvalues, eigenvectors, and their relation to poles of the S-matrix in the second Riemann sheet exhibit close links to the mathematical structure of the Lax–Phillips scattering theory. These relations between the theories suggest that there may exist a framework which combines the favorable properties of the two theories, i.e., the applicability of the RHS construction to a wide range of quantum mechanical resonance scattering problems and the Hilbert space structure of the Lax–Phillips theory. As mentioned in the Introduction, such a framework has been suggested and the consequences of its application to the description of quantum mechanical resonances are explored elsewhere (Strauss and Horwitz, in preparation).

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